New Analytical Solution of the First-Passage Reliability Problem for Linear Oscillators

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Abstract: The classical first-passage reliability problem for linear elastic single-degree-of-freedom (SDOF) oscillators subjected to stationary and nonstationary Gaussian excitations is explored. Several analytical approximations are available in the literature for this problem: the Poisson, classical Vanmarcke, and modified Vanmarcke approximations. These analytical approximations are widely used because of their simplicity and their lower computational cost compared with simulation techniques. However, little is known about their accuracy in estimating the time-variant first-passage failure probability (FPFP) for varying oscillator properties, failure thresholds, and types of loading. In this paper, a new analytical approximation of the FPFP for linear SDOF systems is proposed by modifying the classical Vanmarcke hazard function. This new approximation is verified by comparing its failure probability estimates with the results obtained using existing analytical approximations and the importance sampling using elementary events method for a wide range of oscillator properties, threshold levels, and types of input excitations. It is shown that the newly proposed analytical approximation of the hazard function yields a significantly more accurate estimate of the FPFP compared with the Poisson, classical Vanmarcke, and modified Vanmarcke approximations.
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Introduction

Dynamic engineering systems are characterized by both significant uncertainty in their properties and randomness in their loading environment. Stochastic dynamics is a well-developed and challenging research subject that continues to attract notable interest across many engineering areas. A classical result sought in stochastic dynamics is the failure probability for the first-passage reliability problem, referred to as the first-passage failure probability (FPFP). The FPFP is the probability that a given response quantity of an engineering system subjected to a dynamic stochastic loading outcrosses a specified threshold within a given exposure time.

Many analytical and numerical studies have been devoted to the computation of the FPFP (Rice 1944, 1945; Crandall 1970; Corotis et al. 1972; Vanmarcke 1975; Beck 2008). However, to date, no exact closed-form solution of the FPFP is available even for the simplest case, i.e., a single-degree-of-freedom (SDOF) linear oscillator subjected to Gaussian white noise (WN) excitation with a deterministic failure threshold (Crandall 1970; Naess 1990; Barbato and Conte 2011).

Among the numerical methods used to compute the FPFP, the Monte Carlo simulation is the most general and robust, although it is also computationally expensive (or even prohibitive) for low probability events. Efficient simulation-based methods are available for general nonlinear problems and/or non-Gaussian excitations, such as the average conditional exceedance rate method (Naess and Gaidai 2009; Naess et al. 2010) and the subset simulation method (Au and Beck 2001b; Ching et al. 2005a, b). For linear elastic systems subjected to Gaussian random loading, the importance sampling using elementary events (ISEE) method has been proven to be an extremely efficient simulation method for computing the FPFP (Au and Beck 2001a).

Among the various analytical approximations proposed to estimate the FPFP, the Poisson (P) approximation (Rice 1944, 1945), the classical Vanmarcke (cVM) approximation, and the modified Vanmarcke (mVM) approximation (Corotis et al. 1972; Vanmarcke 1975) are the most widely used. Although these analytical approximations generally provide a good compromise between accuracy and computational effort, their use also presents some deficiencies. For instance, it has been shown that, in the case of stationary processes, the P approximation provides very conservative estimates of the FPFP for low failure thresholds and narrowband processes, whereas the cVM and mVM approximations are characterized by inconsistent levels of accuracy for different structural systems and failure thresholds (Corotis et al. 1972; Vanmarcke 1975). In addition, very little information is available on the accuracy of these analytical approximations in the case of nonstationary processes (Barbato and Conte 2011). In fact, both the cVM and the mVM approximations require computing the so-called bandwidth parameter of the response process of interest. However, only

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recently has an appropriate definition of the time-variant bandwidth parameter of a nonstationary process been given based on the concept of nongeometric spectral characteristics (Di Paola 1985; Michaelov et al. 1999). Currently, the exact closed form of the time-variant bandwidth parameter is available for the displacement response processes of linear SDOF systems. It is also available for displacement response processes (and for any response quantity linearly related to the displacement responses) of classically and nonclassically damped multi-degree-of-freedom systems subjected to Gaussian WN excitation from at-rest initial conditions (Barbato and Conte 2008) and time-modulated nonwhite noise excitation (Barbato and Vasta 2010). These exact solutions have been used to (1) compute the cVM and mVM approximations of the time-variant FPFP for different linear structural systems subjected to different types of loading, and (2) evaluate the absolute and relative accuracy of these analytical approximations compared with the corresponding ISEE results (Barbato and Conte 2011). It was found that the cVM and mVM approximations are, in general, more accurate than the simpler P approximation. However, the relative accuracy of the cVM and mVM approximations can be evaluated only on a case-by-case basis.

This paper aims to (1) investigate the absolute and relative accuracy of existing analytical approximations (i.e., P, cVM, and mVM approximation) of the time-variant FPFP for linear SDOF systems subjected to stationary and/or nonstationary Gaussian excitations modeled as separable nonstationary processes (i.e., stochastic processes defined as the product of a deterministic time modulating function and a stationary process), and (2) derive an improved analytical approximate solution for the time-variant FPFP. The first aim is achieved by evaluating, via an extensive parametric study, the accuracy of the considered analytical estimates of the time-variant FPFP for a wide range of SDOF oscillator properties (i.e., natural periods and damping ratios), failure threshold levels, and seismic input models. This evaluation is based on the comparison of the analytical results with the corresponding simulation results obtained using the ISEE method, which are considered as the reference solution. The second goal is achieved by proposing a new analytical approximation of the time-variant FPFP for linear SDOF systems subjected to stationary and/or nonstationary Gaussian excitations from at-rest initial conditions. The accuracy in estimating the time-variant FPFP of this new analytical approximation is also examined in detail.

Existing Analytical Approximations of the FPFP

The time-variant FPFP, $P_{f,|X|}$, corresponding to the outcrossing of a failure threshold level, $x_{\lim}$, by the absolute value of the random process $X(t)$ (symmetric double-barrier problem) from at-rest initial conditions, is commonly expressed as

$$P_{f,|X|}(x_{\lim},t) = 1 - \exp\left\{-\int_0^t h_{|X|}(x_{\lim},\tau)\cdot \mathrm{d}\tau\right\} \tag{1}$$

where t = time; and $h_{|X|}$ = time-variant hazard function for the symmetric double-barrier problem. In this paper, the process $X(t)$ denotes the displacement response of a structural system subjected to a stationary and/or nonstationary input excitation. To the best of the writers' knowledge, no exact analytical solution is available for $h_{|X|}$ to date. However, several analytical approximations of $h_{|X|}$ are available in the literature. The P, cVM, and mVM approximations are considered and briefly reviewed subsequently.

The P approximation assumes that the number of outcrossing events is described by a Poisson process (i.e., outcrossing events are independent). Thus, the time-variant hazard function is assumed to be equal to the mean outcrossing rate function of process $X(t)$, $\upsilon_{|X|}(x_{\lim},t)$, i.e.

$$h_{\mathrm{P},|X|}(x_{\lim},t) = \upsilon_{|X|}(x_{\lim},t) = \mathrm{E}\left[\frac{\mathrm{d}N(t)}{\mathrm{d}t}\right] \tag{2}$$

where $N(t)$ = number of outcrossing events in the time interval $[0,t]$ and $\mathrm{E}[\ldots]$ = expected value operator. The mean outcrossing rate $\upsilon_{|X|}(x_{\lim},t)$ is known in exact closed form, which is derived by using Rice's formula (Rice 1944, 1945; Lutes and Sarkani 2004). Whereas the P approximation is asymptotically exact for infinite threshold values, its accuracy for threshold levels of practical significance critically depends on the bandwidth of the process (Vanmarcke 1975). In particular, the FPFP is usually overestimated for narrowband processes and low threshold levels, whereas it can be underestimated for wideband processes (Crandall et al. 1966).

The cVM approximation takes into account the effects of the bandwidth of the process, $q_X(t)$ [i.e., it considers the fraction of time that the envelope process spends above or below the failure threshold level and that the outcrossings of the envelope process are not always associated with one or more outcrossings of the actual process (Corotis et al. 1972; Vanmarcke 1975)]. The former consideration is more useful for low failure threshold levels, whereas the latter becomes more important for high failure threshold levels. The cVM approximation assumes the following analytical expression for the time-variant hazard function (Vanmarcke 1975):

$$h_{\mathrm{cVM},|X|}(x_{\lim},t) = \upsilon_{|X|}(x_{\lim},t) \cdot \frac{1-\exp\left\{-\sqrt{\pi/2}\cdot q_X(t)\cdot[x_{\lim}/\sigma_X(t)]\right\}}{1-\exp\left\{-\frac{1}{2}\cdot[x_{\lim}/\sigma_X(t)]^2\right\}} \tag{3}$$

where $\sigma_X(t)$ = time-variant standard deviation of process $X(t)$.

The mVM approximation heuristically accounts for superclumping effects (Corotis et al. 1972; Vanmarcke 1975) by introducing an exponent equal to 1.2 for the time-variant bandwidth parameter in the approximate hazard function equation; i.e.,

$$h_{\mathrm{mVM},|X|}(x_{\lim},t) = \upsilon_{|X|}(x_{\lim},t) \cdot \frac{1-\exp\left\{-\sqrt{\pi/2}\cdot [q_X(t)]^{1.2}\cdot[x_{\lim}/\sigma_X(t)]\right\}}{1-\exp\left\{-\frac{1}{2}\cdot[x_{\lim}/\sigma_X(t)]^2\right\}} \tag{4}$$

The P, cVM, and mVM hazard functions for the single-barrier first-passage problem are obtained from Eqs. (2)–(4), respectively, by substituting $x_{\lim}$ with $x_{\lim}^+$ = threshold for the single-barrier problem, by substituting $\upsilon_{|X|}(x_{\lim},t)$ with $\upsilon_X(x_{\lim}^+,t)$ = mean upcrossing rate of $x_{\lim}^+$ by process $X(t)$, and by substituting $\sqrt{\pi/2}$ with $\sqrt{2\pi}$.

New Analytical Approximation of the FPFP for Linear SDOF Systems

A new analytical hazard function for the displacement response $X(t)$ of a linear SDOF system is proposed to obtain improved estimates of $P_f(t)$ compared with those obtained using the P, cVM, and mVM approximations. First, a new hazard function is derived for the case of linear SDOF systems subjected to WN excitation from at-rest initial conditions. Then, appropriate corrections are derived to account for time modulation and nonwhite spectra of the input loading.

Hazard Function for Linear SDOF Systems Subjected to WN Excitation from At-Rest Initial Conditions

The time-variant FPFP for linear SDOF systems subjected to WN excitation from at-rest initial conditions (i.e., with the unit step time-modulating function) was investigated in Barbato and Conte (2011). It was found that the relative accuracy of the cVM and mVM approximations varies with the damping ratio, exposure time, and threshold level. In addition, for several cases it was noted that the ISEE results shift progressively away from the cVM to the mVM approximation results as time elapses. This observation suggests that a better approximation of the FPFP could be achieved by (1) considering a time-dependent exponent of the time-variant bandwidth parameter that increases with time until it reaches a stationary value, and (2) multiplying the mean outcrossing rate by a time-dependent factor (with values larger than or equal to 1) that decreases with time until it reaches a stationary value equal to 1.

On the basis of these observed trends, a new hazard function for the double-barrier first-passage problem is proposed as follows:

$$h_{\mathrm{New},|X|}(x_{\lim},t)=\upsilon_{|X|}(x_{\lim},t)\cdot\exp[C_2(\xi,\zeta,t)]\cdot\frac{1-\exp\left\{-\sqrt{\pi/2}\cdot[q_X(t)]^{C_1(\xi,\zeta,t)}\cdot[x_{\lim}/\sigma_X(t)]\right\}}{1-\exp\left\{-\frac{1}{2}\cdot[x_{\lim}/\sigma_X(t)]^2\right\}}\tag{5}$$

in which

$$C_1(\xi,\zeta,t)=C_{1,\infty}(\xi,\zeta)\cdot\exp\left\{-\left[q_X(t)-q_{X,\infty}^{(\mathrm{WN})}\right]\right\}\tag{6}$$

$$C_2(\xi,\zeta,t)=C_{2,\infty}(\xi,\zeta)\cdot\left[q_X(t)-q_{X,\infty}^{(\mathrm{WN})}\right]\tag{7}$$

where ξ = damping ratio; $\zeta=x_{\lim}/\sigma_{X,\infty}^{(\mathrm{WN})}=x_{\lim}/\sigma_{X,\max}$ = normalized failure threshold level; $\sigma_{X,\infty}^{(\mathrm{WN})}$ = stationary value of the standard deviation of process $X(t)$ when the input process is a WN time modulated by a unit step function; $\sigma_{X,\max}=\max_{t\geq 0}[\sigma_X(t)]$; $q_{X,\infty}^{(\mathrm{WN})}$ = stationary value of the time-variant bandwidth parameter of process $X(t)$ when the input process is a WN time modulated by a unit step function (Barbato and Conte 2008); and $C_{1,\infty}(\xi,\zeta)$ and $C_{2,\infty}(\xi,\zeta)$ = stationary values of the time-variant functions $C_1(\xi,\zeta,t)$ and $C_2(\xi,\zeta,t)$, respectively. The hazard function for the single-barrier first-passage problem is obtained via the same substitutions described previously for the P, cVM, and mVM approximations.

The time-dependent exponential term in Eq. (6), $\exp\{-[q_X(t)-q_{X,\infty}^{(\mathrm{WN})}]\}$, is introduced to account for the time dependence of the exponent of the time-variant bandwidth parameter (Barbato and Conte 2008) because it always assumes positive values smaller than 1 and increases with time until it reaches a stationary value equal to 1. The stationary part $C_{1,\infty}(\xi,\zeta)$ of function $C_1(\xi,\zeta,t)$ accounts for the dependency on the damping ratio and the normalized failure threshold of the superclumping effects identified in Corotis et al. (1972) and Vanmarcke (1975). The time-dependent term in Eq. (7) reflects the effect of the sudden application of the input loading corresponding to a unit step time-modulating function. It is noteworthy that the time-variant FPFP for a linear SDOF system subjected to WN excitation can be expressed as a function of normalized time $t_0=t/T$ (i.e., the time t divided by the natural period T of the SDOF system; see the appendix). Thus, the stationary values $C_{1,\infty}(\xi,\zeta)$ and $C_{2,\infty}(\xi,\zeta)$ are independent of T.

A closed-form expression for $C_1(\xi,\zeta,t)$ and $C_2(\xi,\zeta,t)$ is obtained by deriving an analytical expression for the stationary values $C_{1,\infty}(\xi,\zeta)$ and $C_{2,\infty}(\xi,\zeta)$ as follows. First, the ISEE method is repeatedly applied to evaluate the time history of the time-variant FPFP, $P_f(t)$, for different combinations of the damping ratio ($\bar{\xi}=0.01$, 0.02, 0.05, 0.10, 0.20, 0.30, 0.40, and 0.50) and of the normalized failure threshold level ($\bar{\zeta}=1.5$, 2, 3, 4, and 5). Closely spaced values of t_0 are considered from $t_0=0$ to a value of t_0 sufficiently large to reach stationarity. For each of these $8\times 5=40$ time histories, the values of $\bar{C}_{1,\infty}(\bar{\xi},\bar{\zeta})$ and $\bar{C}_{2,\infty}(\bar{\xi},\bar{\zeta})$ are obtained through least-squares fitting using the MATLAB function "lsqcurvefit" (MathWorks 1997). These values correspond to the best fit between the analytical estimates of $P_f(t_0)$, computed using Eqs. (1) and (5), and the ISEE results. The ISEE results used to estimate $\bar{C}_{1,\infty}(\bar{\xi},\bar{\zeta})$ and $\bar{C}_{2,\infty}(\bar{\xi},\bar{\zeta})$ are obtained by imposing a very low target coefficient of variation (COV = 0.001), to ensure high accuracy in the estimates of $P_f(t_0)$ and to minimize the sensitivity of $\bar{C}_{1,\infty}(\bar{\xi},\bar{\zeta})$ and $\bar{C}_{2,\infty}(\bar{\xi},\bar{\zeta})$ to the variability of the simulation results. Fig. 1 compares the values of $P_f(t_0)$ computed using (1) the ISEE method; (2) the P, cVM, and mVM approximations of the hazard function; and (3) the newly proposed analytical approximation of the hazard function given in Eq. (5) and based on the fitted values of $\bar{C}_{1,\infty}(\bar{\xi},\bar{\zeta})$ and $\bar{C}_{2,\infty}(\bar{\xi},\bar{\zeta})$, for the case corresponding to $T=0.1$ s, $\xi=0.2$, and $\zeta=3.0$. The hazard function built using the optimized values $\bar{C}_{1,\infty}(\bar{\xi},\bar{\zeta})$ and $\bar{C}_{2,\infty}(\bar{\xi},\bar{\zeta})$ is able to reproduce the simulation results with extremely high accuracy over the entire time history of $P_f(t_0)$. Similar results are obtained for all combinations of $\bar{\xi}$ and $\bar{\zeta}$ considered here, thus validating the analytical expressions assumed for the time-variant hazard function in Eqs. (5)–(7).

Two polynomial surfaces $C_{1,\infty}(\xi,\zeta)$ and $C_{2,\infty}(\xi,\zeta)$ (expressed as functions of the damping ratio ξ and of the normalized threshold ζ) are then fitted to the values $\bar{C}_{1,\infty}(\bar{\xi},\bar{\zeta})$ and $\bar{C}_{2,\infty}(\bar{\xi},\bar{\zeta})$ obtained via least-squares fitting by using the "sftool" MATLAB toolbox (MathWorks 1997). This fitting is done to extend the proposed expression of the hazard function to values of ξ and ζ for which simulation results are not available. The order of the polynomials is kept as small as possible (equal to 5 for the damping ratio and 4 for the normalized threshold) to balance the contrasting requirements of accuracy and simplicity. The following polynomial representations are suggested for $C_{1,\infty}(\xi,\zeta)$ and $C_{2,\infty}(\xi,\zeta)$:

$$C_{i,\infty}(\xi,\zeta)=\sum_{l=0}^{5}\sum_{m=0}^{4}(P_{lm}^{(i)}\cdot\xi^l\cdot\zeta^m);\qquad i=1,2;\qquad 0.01\leq\xi\leq 0.50;\qquad 1.5\leq\zeta\leq 5.0\tag{8}$$

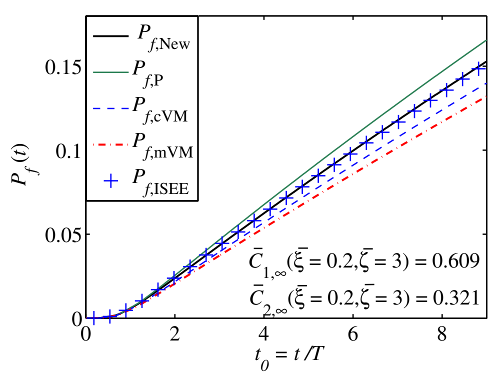

Fig. 1. Computation of $\bar{C}_{1,\infty}$ and $\bar{C}_{2,\infty}$: SDOF system subjected to WN base excitation from at-rest initial conditions ($\bar{\xi}=0.2$, $\bar{\zeta}=3$)

Table 1. Coefficients of the Polynomial Representation of $C_{i,\infty}(i = 1, 2)$ Given in Eq. (8)

i	$P_{00}^{(i)}$	$P_{10}^{(i)}$	$P_{01}^{(i)}$	$P_{20}^{(i)}$	$P_{11}^{(i)}$	$P_{02}^{(i)}$	$P_{30}^{(i)}$	$P_{21}^{(i)}$	$P_{12}^{(i)}$	$P_{03}^{(i)}$	$P_{40}^{(i)}$	$P_{31}^{(i)}$	$P_{22}^{(i)}$	$P_{13}^{(i)}$	$P_{04}^{(i)}$	$P_{50}^{(i)}$	$P_{41}^{(i)}$	$P_{32}^{(i)}$	$P_{23}^{(i)}$	$P_{14}^{(i)}$
1	1.566	−22.23	−0.091	99.95	16.26	0.07	−252.3	−52.57	−5.714	−0.027	367.9	67.92	10.16	0.88	0.003	−214	−39.94	−4.476	−0.654	−0.052
2	−5.319	29.07	6.734	39.89	−48.47	−1.639	−711.7	162	7.3	0.12	1712	−181.5	−23.96	0.192	0.001	−1304	80.33	11.93	1.218	−0.081

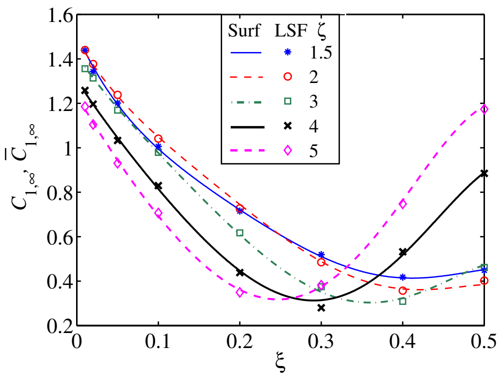

Fig. 2. Comparison between the interpolation surface $C_{1,\infty}$ (Surf) and values $\bar{C}_{1,\infty}$ obtained through least-square fitting (LSF)

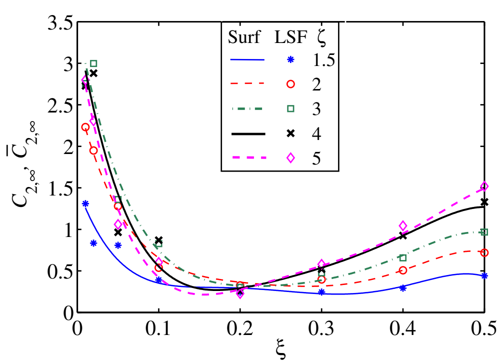

Fig. 3. Comparison between the interpolation surface $C_{2,\infty}$ (Surf) and values $\bar{C}_{2,\infty}$ obtained through least-square fitting (LSF)

in which the coefficients $P_{lm}^{(i)}$ ($i = 1, 2$; $l = 0, 1, 2, 3, 4, 5$; and $m = 0, 1, 2, 3, 4$) obtained from the surface fitting are given in Table 1. For values of the damping ratio ξ and of the normalized threshold ζ that are outside the domain in which the two surfaces previously described are fitted, Eq. (8) is computed using the values of ξ and ζ along the boundaries of the fitting domain [e.g., for damping ratios smaller than 0.01, Eq. (8) is computed using $\xi = 0.01$]. Figs. 2 and 3 plot the isocurves of $C_{1,\infty}(\xi, \zeta)$ and $C_{2,\infty}(\xi, \zeta)$, respectively, for discrete values of ζ as functions of ξ. These polynomial surfaces provide an accurate fit to the values $\bar{C}_{1,\infty}(\bar{\xi}, \bar{\zeta})$ and $\bar{C}_{2,\infty}(\bar{\xi}, \bar{\zeta})$ obtained through least-squares fitting.

Modifications of the Hazard Function Needed to Account for Time Modulation and Nonwhite Spectra

When the input process is time modulated [with the time-modulating function $A(t)$] and/or characterized by a nonwhite spectrum (i.e., the excitation is a colored noise), the results obtained considering SDOF systems subjected to WN excitation from at-rest initial conditions still can be used by modifying Eqs. (6) and (7) through the following corrections:

Correction 1. The functions in Eqs. (6) and (7) are computed using an equivalent damping ratio, $\tilde{\xi}$, and an equivalent time-variant normalized threshold level, $\tilde{\zeta}(t)$, i.e.,

$$C_1(\tilde{\xi}, \tilde{\zeta}(t), t) = C_{1,\infty}(\tilde{\xi}, \tilde{\zeta}(t)) \cdot \exp\{\min\{-[q_X(t) - q_{X,\infty}], 0\}\} \tag{9}$$

$$C_2(\tilde{\xi}, \tilde{\zeta}(t), t) = C_{2,\infty}(\tilde{\xi}, \tilde{\zeta}(t)) \cdot \max\{[q_X(t) - q_{X,\infty}], 0\} \tag{10}$$

in which $q_{X,\infty}$ = stationary value of the time-variant bandwidth parameter of process $X(t)$ computed using the unit step function as the time-modulating function. The min and max operators are introduced because the quantity $[q_X(t) - q_{X,\infty}]$ can become negative for general nonstationary excitations, whereas the quantity $[q_X(t) - q_{X,\infty}^{(\mathrm{WN})}]$ is always positive for WN excitation and at-rest initial conditions.

Correction 2. The equivalent damping ratio, $\tilde{\xi}$, is taken as the value of the damping ratio that would provide the same stationary value of the time-variant bandwidth parameter obtained from the colored noise excitation if the SDOF system was subjected to WN excitation. This equivalent damping ratio can be found using the closed-form expression of the time-variant bandwidth parameter (Barbato and Conte 2008) as follows:

$$\tilde{\xi} = \mathrm{zero}\left\{ q_{X,\infty} - \left\{ 1 - \frac{4 \cdot \left[\arctan\left(\sqrt{1-\xi^2}/\xi\right)\right]^2}{\pi^2 \cdot (1-\xi^2)} \right\}^{1/2} \right\} \tag{11}$$

in which the operator zero = root of the quantity in parentheses. Because the time-variant bandwidth parameter is a monotonically increasing continuous function of ξ, the solution $\tilde{\xi}$ of the transcendental Eq. (11) exists and is unique for any $0 \leq q_{X,\infty} < 1$.

Correction 3. The equivalent normalized failure threshold, $\tilde{\zeta}(t)$, is computed as follows:

$$\tilde{\zeta}(t) = \frac{x_{\lim}}{\sigma_{X,\max} \cdot A(t)} \cdot \frac{G(\omega_0)}{S_0} \tag{12}$$

in which $G(\omega_0)$ = power spectral density of the colored noise excitation computed at $\omega_0 = 2 \cdot \pi / T$ = natural circular frequency of the SDOF system and $S_0 = 1 \cdot \mathrm{m}^2/\mathrm{s}^3$ = normalization factor.

Correction 4. If $A(t)$ does not present a discontinuity for $t = 0$ s (i.e., the time-modulating function increases gradually from zero to its maximum value), it is assumed that $C_2(\tilde{\xi}, \tilde{\zeta}(t), t) = 0$. Otherwise, $C_2(\tilde{\xi}, \tilde{\zeta}(t), t)$ is computed according to Eq. (10).

The time-modulating function, $A(t)$, is scaled here so that its maximum value in time is equal to 1 [i.e., $\max_{t \geq 0}[A(t)] = 1$ (see Fig. 4)]. It is noteworthy that the equivalent damping ratio $\tilde{\xi}$ and the equivalent normalized threshold level $\tilde{\zeta}(t)$ reduce to the actual damping ratio ξ and normalized threshold level ζ for the case of SDOF systems subjected to WN excitation from at-rest initial conditions, i.e., Eqs. (6) and (7) can be regarded as special cases of Eqs. (9) and (10).

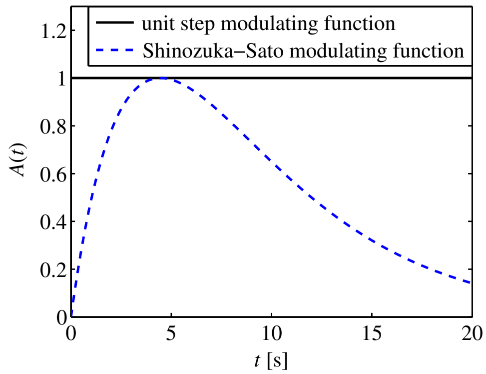

Fig. 4. Time modulating functions: (1) unit step function; and (2) Shinozuka-Sato with $B_1 = 0.20$ and $B_2 = 0.25$

Parametric Study to Evaluate the Accuracy of the Existing and Newly Proposed Analytical Approximations

The results of an extensive parametric study are presented here to evaluate the absolute and relative accuracy of the P, cVM, and mVM approximations, as well as the accuracy of the newly proposed analytical approximation (denoted as New). This parametric study considers a wide range of natural periods, damping ratios, and threshold levels, as well as different input spectra and time-modulating functions. The time-variant FPFP results obtained from the analytical approximations are compared with the corresponding results obtained using the ISEE method with a target COV of 0.01, which is considered the reference solution. The value of COV = 0.01 used for this parametric study is chosen because it provides sufficiently accurate estimates of the FPFP at a feasible computational cost (several times smaller than the computational effort required to obtain the value COV = 0.001 used to calibrate the analytical approximation denoted as New). The comparison between analytical approximations and ISEE results is based on the percent error $\varepsilon = 100 \cdot (P_{f,i} - P_{f,\mathrm{ISEE}})/P_{f,\mathrm{ISEE}}$ (i = P, cVM, mVM, New). For each type of input loading considered, synoptic results are also provided including the maximum ($\varepsilon_{\max}$) and minimum ($\varepsilon_{\min}$) percent errors, as well as the mean of the absolute values of the percent errors ($\mu_{|\varepsilon|}$) computed for the different damping ratios and normalized threshold levels considered. These synoptic results indicate whether a given approximate analytical solution tends to overestimate ($\varepsilon_{\max}$) or to underestimate ($\varepsilon_{\min}$) the FPFP. They also provide information regarding the average value of the error corresponding to a specific analytical solution for a given loading condition ($\mu_{|\varepsilon|}$). Because of space constraints, only selected results are presented in this paper. Additional results can be found in Ghazizadeh (2011). It is worth mentioning that the different analytical approximations involve a similar computational cost, which can be several orders of magnitude smaller than the computational cost associated with the ISEE method.

Input Excitation Models

Two types of random excitations are considered in this study: (1) a WN excitation, represented by a constant power spectral density [i.e., $G(\omega) = S_0$, in which ω = circular frequency, and $S_0 = 1\ \mathrm{m}^2/\mathrm{s}^3$], and (2) a nonwhite excitation, modeled by using a Kanai-Tajimi (KT) power spectral density function. The KT power spectral density function is given by

$$G(\omega) = \frac{1 + 4 \cdot \xi_g^2 \cdot (\omega/\omega_g)^2}{[1 - (\omega/\omega_g)^2]^2 + (2 \cdot \xi_g \cdot \omega/\omega_g)^2} \cdot S_0 \tag{13}$$

in which ξ_g = predominant ground damping ratio and ω_g = ground natural circular frequency. In this study, $\xi_g = 0.6$ and $\omega_g = 9\pi$ are used (Vanmarcke 1976).

Two time-modulating functions are considered: (1) a unit step function, $H(t)$, and (2) a Shinozuka-Sato modulating function (Shinozuka and Sato 1967). The latter modulating function is defined as

$$A(t) = C \cdot [e^{-B_1 t} - e^{-B_2 t}] \cdot H(t) \tag{14}$$

in which B_1 and B_2 = constants defining the shape of the time-modulating function and C = normalizing constant. In this study, the following values are assumed for these constants: $B_1 = 0.20$, $B_2 = 0.25$, and $C = 12.207$. The two modulating functions considered are shown in Fig. 4.

Table 2. Time-Variant FPFP ($t_0 = 10$) for Linear Elastic SDOF Systems Subjected to WN Base Excitation from At-Rest Initial Conditions (i.e., with Unit Step Time-Modulating Function)

ζ	ξ	ISEE	P	ε (%)	cVM	ε (%)	mVM	ε (%)	New	ε (%)
2	0.01	1.23E − 01	3.39E − 01	176.24	1.66E − 01	35.26	1.29E − 01	4.87	1.20E − 01	−2.29
	0.05	6.15E − 01	8.68E − 01	41.22	6.78E − 01	10.25	6.07E − 01	−1.37	6.11E − 01	−0.59
	0.10	7.84E − 01	9.07E − 01	15.77	7.98E − 01	1.85	7.52E − 01	−4.03	7.92E − 01	1.07
	0.50	9.32E − 01	9.29E − 01	−0.35	9.13E − 01	−2.04	9.05E − 01	−2.90	9.33E − 01	0.04
3	0.01	3.49E − 03	7.66E − 03	119.47	4.07E − 03	16.64	3.14E − 03	−10.01	3.20E − 03	−8.35
	0.05	7.79E − 02	1.34E − 01	71.73	8.67E − 02	11.26	7.35E − 02	−5.60	7.71E − 02	−1.05
	0.10	1.28E − 01	1.68E − 01	31.41	1.26E − 01	−0.96	1.14E − 01	−10.83	1.29E − 01	0.82
	0.50	1.91E − 01	1.93E − 01	1.05	1.79E − 01	−6.22	1.76E − 01	−8.13	1.88E − 01	−2.00
4	0.01	2.24E − 05	3.82E − 05	70.60	2.37E − 05	5.55	1.86E − 05	−16.96	2.15E − 05	−4.19
	0.05	2.72E − 03	3.79E − 03	39.36	2.75E − 03	0.89	2.37E − 03	−13.03	2.70E − 03	−0.58
	0.10	4.60E − 03	5.25E − 03	14.04	4.31E − 03	−6.23	3.94E − 03	−14.25	4.62E − 03	0.47
	0.50	6.31E − 03	6.41E − 03	0.02	6.16E − 03	−2.36	6.08E − 03	−3.65	6.21E − 03	−1.43
			$\varepsilon_{\max}$%	176.24	$\varepsilon_{\max}$%	35.26	$\varepsilon_{\max}$%	4.87	$\varepsilon_{\max}$%	1.07
			$\varepsilon_{\min}$%	−0.35	$\varepsilon_{\min}$%	−6.23	$\varepsilon_{\min}$%	−16.96	$\varepsilon_{\min}$%	−8.35
			$\mu_{|\varepsilon|}$%	48.44	$\mu_{|\varepsilon|}$%	8.29	$\mu_{|\varepsilon|}$%	7.97	$\mu_{|\varepsilon|}$%	1.90

Table 3. Time-Variant FPFP for Linear Elastic SDOF Systems Subjected to WN Base Excitation from At-Rest Initial Conditions (i.e., with Unit Step Time-Modulating Function) for a Given Normalized Threshold of $\zeta = 5$ and Different Values of the Normalized Time t_0

t_0	ξ	ISEE	P	ε (%)	cVM	ε (%)	mVM	ε (%)	New	ε (%)
10	0.01	3.53E − 08	5.00E − 08	41.56	3.46E − 08	−2.12	2.78E − 08	−21.38	3.42E − 08	−3.27
	0.05	3.16E − 05	3.74E − 05	18.20	2.98E − 05	−5.91	2.62E − 05	−17.18	3.10E − 05	−2.07
	0.10	5.28E − 05	5.59E − 05	5.97	4.94E − 05	−6.44	4.60E − 05	−12.82	5.32E − 05	0.75
	0.50	6.97E − 05	7.09E − 05	1.83	6.97E − 05	−0.02	6.91E − 05	−0.77	6.93E − 05	−0.48
20	0.01	4.69E − 06	9.74E − 06	107.75	5.57E − 06	18.78	4.18E − 06	−10.74	4.66E − 06	−0.63
	0.05	9.29E − 05	1.12E − 04	20.17	8.81E − 05	−5.18	7.72E − 05	−16.86	9.12E − 05	−1.82
	0.10	1.24E − 04	1.30E − 04	5.59	1.15E − 04	−6.94	1.07E − 04	−13.36	1.24E − 04	0.23
	0.50	1.44E − 04	1.45E − 04	0.83	1.43E − 04	−1.01	1.42E − 04	−1.76	1.42E − 04	−1.57
50	0.01	7.01E − 05	1.86E − 04	165.23	9.60E − 05	37.03	6.99E − 05	−0.26	7.40E − 05	5.61
	0.05	2.75E − 04	3.35E − 04	22.10	2.64E − 04	−3.95	2.31E − 04	−15.90	2.73E − 04	−0.73
	0.10	3.37E − 04	3.54E − 04	4.99	3.12E − 04	−7.55	2.90E − 04	−13.97	3.36E − 04	−0.42
	0.50	3.72E − 04	3.69E − 04	−0.78	3.62E − 04	−2.60	3.60E − 04	−3.34	3.60E − 04	−3.23
			ε_{max}%	165.23	ε_{max}%	37.03	ε_{max}%	−0.26	ε_{max}%	5.61
			ε_{min}%	−0.78	ε_{min}%	−7.55	ε_{min}%	−21.38	ε_{min}%	−3.27
			$\mu_{\|\varepsilon\|}$%	32.92	$\mu_{\|\varepsilon\|}$%	8.13	$\mu_{\|\varepsilon\|}$%	10.70	$\mu_{\|\varepsilon\|}$%	1.73

Results for Linear SDOF Systems Subjected to WN Excitation from At-Rest Initial Conditions

For linear SDOF systems subjected to WN excitation from at-rest initial conditions, it can be shown that the time-variant FPFP depends on the time t and the natural period of the system T only through the normalized time $t_0 = t/T$ (see the appendix). Thus, the results are presented here as a function of the normalized time, damping ratio, and normalized threshold, and are valid for any natural period of the system. Table 2 shows the estimates of the FPFP corresponding to $t_0 = 10$; normalized thresholds $\zeta = 2$, 3, and 4; and damping ratios $\xi = 0.01$, 0.05, 0.10, and 0.50, computed using the ISEE method, as well as using the P, cVM, mVM, and New approximations. The FPFP values range from a maximum value of 0.932 to a minimum value of 2.24×10^{-5}. Table 2 also provides the percent error ε, as well as the minimum percent error, maximum percent error, and mean of the absolute values of the percent error for each of the considered analytical approximations. As expected, the P approximation consistently overestimates the FPFP, sometimes even by a large factor. The cVM and mVM approximations produce similar results and are more accurate than the P approximation. The cVM approximation tends to overestimate the FPFP, particularly for small damping, whereas the mVM approximation tends to underestimate the FPFP, particularly for large thresholds. The New approximation is overall significantly more accurate than all other analytical approximations, with a slight tendency to underestimate the FPFP. It presents the smallest value of the maximum error ($\varepsilon_{max} = 1.07\%$) and of the mean of the absolute values of the errors ($\mu_{|\varepsilon|} = 1.90\%$), which are more than four times smaller than the corresponding quantities for the second best approximation (i.e., in this case, the mVM approximation).

Table 3 provides the FPFP estimates for linear SDOF systems subjected to WN excitation from at-rest initial conditions with given normalized threshold $\zeta = 5$ and varying normalized time $t_0 = 10$, 20, and 50. The FPFP values ranged from a maximum value of 3.72×10^{-4} to a minimum value of 3.53×10^{-8}. Also in this case, it is observed that the P approximation overestimates the FPFP, the cVM approximation tends to overestimate the FPFP, and the mVM approximation tends to underestimate the FPFP. The New approximation is very accurate under all combinations of damping ratios and normalized times, with $\mu_{|\varepsilon|} = 1.73\%$.

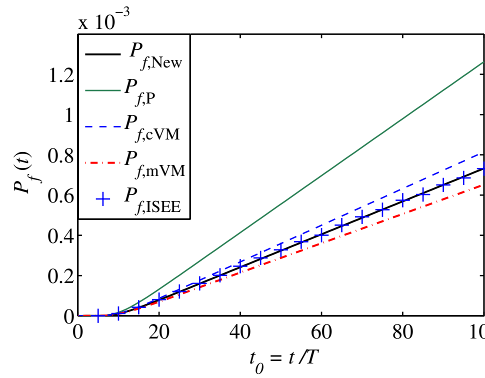

Fig. 5. FPFP for a linear elastic SDOF system subjected to WN base excitation from at-rest initial conditions ($\xi = 0.023$, $\zeta = 4.87$)

Fig. 5 plots the comparison of the FPFP estimates obtained using the ISEE method and the P, cVM, mVM, and New approximations for a linear SDOF system with damping ratio $\xi = 0.023$ for a normalized threshold of $\zeta = 4.87$. It is noteworthy that the damping ratio and the normalized threshold values used in this example are not included among the values used to define the functions $C_{1,\infty}(\xi, \zeta)$ and $C_{2,\infty}(\xi, \zeta)$. The agreement between the results obtained using the ISEE method and the New approximation is excellent over the entire range of normalized times, clearly showing that the New approximation is superior to all other considered analytical approximations for this input loading case.

Results for Linear SDOF Systems Subjected to WN Excitation Modulated in Time by a Shinozuka-Sato Function

For linear SDOF systems subjected to time-modulated WN excitation, the time-variant FPFP depends on both the time, t, and the natural period of the system, T. Table 4 presents the FPFP values obtained using the ISEE method and the four analytical

Table 4. Time-Variant FPFP Computed at $t = 20$ s for Linear Elastic SDOF Systems with the Natural Period $T = 1.0$ s Subjected to WN Base Excitation Time Modulated by a Shinozuka-Sato Function

ζ	ξ	ISEE	P	ε (%)	cVM	ε (%)	mVM	ε (%)	New	ε (%)
2	0.01	3.90E − 01	9.21E − 01	136.15	5.00E − 01	28.16	3.77E − 01	−3.38	3.21E − 01	−17.62
	0.05	4.99E − 01	7.83E − 01	56.83	5.61E − 01	12.37	4.88E − 01	−2.24	4.87E − 01	−2.48
	0.10	5.95E − 01	7.53E − 01	26.53	6.07E − 01	2.02	5.56E − 01	−6.47	5.99E − 01	0.69
	0.50	7.50E − 01	7.41E − 01	−1.23	7.09E − 01	−5.43	6.97E − 01	−7.11	7.40E − 01	−1.30
3	0.01	4.01E − 02	1.37E − 01	240.73	4.94E − 02	23.33	3.46E − 02	−13.59	3.36E − 02	−16.12
	0.05	4.95E − 02	8.47E − 02	71.03	5.36E − 02	8.21	4.50E − 02	−9.12	4.84E − 02	−2.38
	0.10	5.87E − 02	7.76E − 02	32.32	5.81E − 02	−0.91	5.21E − 02	−11.15	5.99E − 02	2.07
	0.50	7.43E − 02	7.52E − 02	1.27	6.98E − 02	−6.00	6.84E − 02	−7.93	7.25E − 02	−2.33
4	0.01	1.33E − 03	3.38E − 03	154.94	1.45E − 03	9.36	1.03E − 03	−22.39	1.09E − 03	−17.68
	0.05	1.51E − 03	2.05E − 03	36.22	1.47E − 03	−2.32	1.26E − 03	−16.25	1.48E − 03	−1.95
	0.10	1.65E − 03	1.87E − 03	13.42	1.55E − 03	−6.40	1.41E − 03	−14.41	1.67E − 03	0.90
	0.50	1.80E − 03	1.81E − 03	0.62	1.74E − 03	−3.12	1.72E − 03	−4.35	1.75E − 03	−2.78
			ε_{max}%	240.73	ε_{max}%	28.16	ε_{max}%	−2.24	ε_{max}%	2.07
			ε_{min}%	−1.23	ε_{min}%	−6.40	ε_{min}%	−22.39	ε_{min}%	−17.68
			$\mu_{\lvert\varepsilon\rvert}$%	64.27	$\mu_{\lvert\varepsilon\rvert}$%	8.97	$\mu_{\lvert\varepsilon\rvert}$%	9.87	$\mu_{\lvert\varepsilon\rvert}$%	5.69

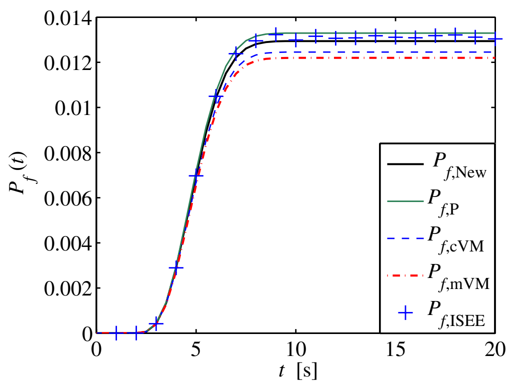

Fig. 6. FPFP for a linear elastic SDOF system with natural period $T = 1.0$ s subjected to WN base excitation with a Shinozuka-Sato modulating function ($\xi = 0.42$, $\zeta = 3.5$)

approximations considered for SDOF oscillators with $T = 1.0$ s and with different damping ratios at $t = 20$ s. The FPFP values ranged from a maximum value of 0.750 to a minimum value of 1.33×10^{-3}. Also, in this case, the P approximation largely overestimates the FPFP, the cVM approximation tends to overestimate the FPFP (particularly for small damping ratios), and the mVM approximation underestimates the FPFP (particularly for high thresholds). The New approximation is the most accurate among the analytical approximations (with $\mu_{|\varepsilon|} = 5.69\%$), even if it underestimates the FPFP for small damping ratios.

Fig. 6 plots the comparison of the FPFP estimates obtained using the ISEE method and the P, cVM, mVM, and New approximations for a linear SDOF system with $T = 1.0$ s and $\xi = 0.42$ for a normalized threshold value of $\zeta = 3.5$. The New approximation provides the best agreement with the ISEE results among all analytical approximations considered here.

Table 5. Time-Variant FPFP Computed at $t = 1.0$ s for a Linear Elastic SDOF System with Natural Period $T = 0.1$ s Subjected to KT Base Excitation from At-Rest Initial Conditions (i.e., with Unit Step Time-Modulating Function)

ζ	ξ	ISEE	P	ε (%)	cVM	ε (%)	mVM	ε (%)	New	ε (%)
2	0.01	1.60E − 01	3.74E − 01	133.71	2.18E − 01	36.11	1.79E − 01	12.05	1.78E − 01	11.16
	0.05	6.84E − 01	8.55E − 01	25.04	7.06E − 01	3.26	6.50E − 01	−5.01	6.92E − 01	1.12
	0.10	8.05E − 01	8.72E − 01	8.31	7.71E − 01	−4.22	7.30E − 01	−9.25	7.90E − 01	−1.86
	0.50	7.97E − 01	7.84E − 01	−1.56	7.32E − 01	−8.17	7.10E − 01	−10.88	7.75E − 01	−2.71
3	0.01	5.02E − 03	9.68E − 03	93.05	6.00E − 03	19.73	4.92E − 03	−1.89	4.75E − 03	−5.33
	0.05	9.30E − 02	1.30E − 01	39.45	9.27E − 02	−0.27	8.18E − 02	−12.05	8.93E − 02	−4.00
	0.10	1.29E − 01	1.49E − 01	15.32	1.17E − 01	−9.19	1.07E − 01	−16.74	1.22E − 01	−5.61
	0.50	1.12E − 01	1.17E − 01	4.40	1.04E − 01	−7.68	9.98E − 02	−11.23	1.12E − 01	−0.76
4	0.01	3.44E − 05	5.60E − 05	62.90	3.96E − 05	15.26	3.31E − 05	−3.62	3.17E − 05	−7.69
	0.05	3.13E − 03	3.71E − 03	18.32	2.92E − 03	−6.72	2.62E − 03	−16.41	2.82E − 03	−10.00
	0.10	4.35E − 03	4.65E − 03	6.90	3.97E − 03	−8.72	3.69E − 03	−14.98	4.09E − 03	−5.97
	0.50	3.56E − 03	3.75E − 03	5.37	3.49E − 03	−1.72	3.40E − 03	−4.35	3.65E − 03	2.78
			ε_{max}%	133.71	ε_{max}%	36.11	ε_{max}%	12.05	ε_{max}%	11.16
			ε_{min}%	−1.56	ε_{min}%	−9.19	ε_{min}%	−16.74	ε_{min}%	−10.00
			$\mu_{\lvert\varepsilon\rvert}$%	34.53	$\mu_{\lvert\varepsilon\rvert}$%	10.09	$\mu_{\lvert\varepsilon\rvert}$%	9.87	$\mu_{\lvert\varepsilon\rvert}$%	4.91

Table 6. Time-Variant FPFP Computed at $t = 5.0$ s for a Linear Elastic SDOF System with Natural Period $T = 0.5$ s Subjected to KT Base Excitation from At-Rest Initial Conditions (i.e., with Unit Step Time-Modulating Function)

ζ	ξ	ISEE	P	ε (%)	cVM	ε (%)	mVM	ε (%)	New	ε (%)
2	0.01	1.20E − 01	3.39E − 01	182.15	1.49E − 01	24.35	1.12E − 01	−6.92	1.09E − 01	−9.49
	0.05	5.94E − 01	8.69E − 01	46.43	6.39E − 01	7.66	5.56E − 01	−6.37	6.31E − 01	6.30
	0.10	7.91E − 01	9.09E − 01	14.92	7.68E − 01	−2.92	7.09E − 01	−10.40	7.41E − 01	−6.29
	0.50	9.23E − 01	9.29E − 01	0.63	8.99E − 01	−2.61	8.85E − 01	−4.09	9.31E − 01	0.80
3	0.01	3.34E − 03	7.66E − 03	129.25	3.67E − 03	9.85	2.73E − 03	−18.30	3.20E − 03	−4.38
	0.05	7.56E − 02	1.34E − 01	77.18	7.94E − 02	4.97	6.51E − 02	−13.87	8.89E − 02	17.49
	0.10	1.26E − 01	1.69E − 01	34.14	1.18E − 01	−6.12	1.03E − 01	−17.93	1.25E − 01	−0.92
	0.50	1.92E − 01	1.93E − 01	0.75	1.73E − 01	−9.80	1.67E − 01	−12.88	1.87E − 01	−2.32
4	0.01	2.21E − 05	3.83E − 05	73.04	2.16E − 05	−2.46	1.64E − 05	−26.13	2.02E − 05	−8.59
	0.05	2.65E − 03	3.80E − 03	43.07	2.54E − 03	−4.31	2.12E − 03	−20.28	2.97E − 03	12.02
	0.10	4.48E − 03	5.28E − 03	18.06	4.08E − 03	−8.80	3.63E − 03	−19.00	4.44E − 03	−0.71
	0.50	6.33E − 03	6.41E − 03	1.37	6.01E − 03	−4.96	5.87E − 03	−7.27	6.29E − 03	−0.55
			ε_{max}%	182.15	ε_{max}%	24.35	ε_{max}%	−4.09	ε_{max}%	17.49
			ε_{min}%	0.63	ε_{min}%	−9.80	ε_{min}%	−26.13	ε_{min}%	−9.49
			$\mu_{\vert\varepsilon\vert}$%	51.75	$\mu_{\vert\varepsilon\vert}$%	7.40	$\mu_{\vert\varepsilon\vert}$%	13.62	$\mu_{\vert\varepsilon\vert}$%	5.82

Table 7. Time-Variant FPFP Computed at $t = 10.0$ s for a Linear Elastic SDOF System with Natural Period $T = 1.0$ s Subjected to KT Base Excitation from At-Rest Initial Conditions (i.e., with Unit Step Time-Modulating Function)

ζ	ξ	ISEE	P	ε (%)	cVM	ε (%)	mVM	ε (%)	New	ε (%)
2	0.01	1.22E − 01	3.42E − 01	180.89	1.66E − 01	36.11	1.28E − 01	5.15	1.21E − 01	−0.56
	0.05	6.12E − 01	8.71E − 01	42.32	6.76E − 01	10.55	6.03E − 01	−1.37	6.06E − 01	−0.95
	0.10	8.00E − 01	9.10E − 01	13.76	7.98E − 01	−0.23	7.51E − 01	−6.17	7.89E − 01	−1.47
	0.50	9.50E − 01	9.36E − 01	−1.46	9.19E − 01	−3.34	9.10E − 01	−4.27	9.40E − 01	−1.09
3	0.01	3.55E − 03	7.80E − 03	119.71	4.10E − 03	15.43	3.15E − 03	−11.30	3.32E − 03	−6.41
	0.05	7.83E − 02	1.35E − 01	72.10	8.64E − 02	10.35	7.30E − 02	−6.75	7.85E − 02	0.16
	0.10	1.31E − 01	1.70E − 01	29.23	1.27E − 01	−3.54	1.14E − 01	−13.50	1.31E − 01	−0.53
	0.50	1.99E − 01	2.00E − 01	0.69	1.85E − 01	−7.30	1.80E − 01	−9.47	1.94E − 01	−2.42
4	0.01	2.32E − 05	3.94E − 05	69.97	2.41E − 05	4.08	1.89E − 05	−18.46	2.27E − 05	−2.11
	0.05	2.71E − 03	3.82E − 03	41.17	2.74E − 03	1.26	2.35E − 03	−13.07	2.78E − 03	2.51
	0.10	4.67E − 03	5.32E − 03	14.03	4.34E − 03	−7.02	3.95E − 03	−15.30	4.72E − 03	1.09
	0.50	6.65E − 03	6.68E − 03	0.38	6.38E − 03	−4.13	6.28E − 03	−5.64	6.46E − 03	−2.82
			ε_{max}%	180.89	ε_{max}%	36.11	ε_{max}%	5.15	ε_{max}%	2.51
			ε_{min}%	−1.46	ε_{min}%	−7.30	ε_{min}%	−18.46	ε_{min}%	−6.41
			$\mu_{\vert\varepsilon\vert}$%	48.81	$\mu_{\vert\varepsilon\vert}$%	8.61	$\mu_{\vert\varepsilon\vert}$%	9.20	$\mu_{\vert\varepsilon\vert}$%	1.84

Results for Linear SDOF Systems Subjected to a KT Excitation from At-Rest Initial Conditions

For linear SDOF systems subjected to nonwhite excitation, the time-variant FPFP depends on the time, t, and the natural period of the system, T. Tables 5–7 compare the FPFP values computed using the ISEE method and the four analytical approximations considered for linear SDOF systems with $T = 0.1$, 0.5, and 1.0 s, respectively, subjected to a KT excitation from at-rest initial conditions. The following observations are made: (1) the P approximation largely overestimates the FPFP, with very large errors for small values of the damping ratio; (2) the cVM approximation tends to overestimate the FPFP, particularly for low damping ratios and low thresholds; and (3) the mVM tends to underestimate the FPFP. Again, the New approximation is more accurate than the other analytical approximations, particularly in the case of $T = 1.0$ s, in which the New approximation presents an accuracy similar to

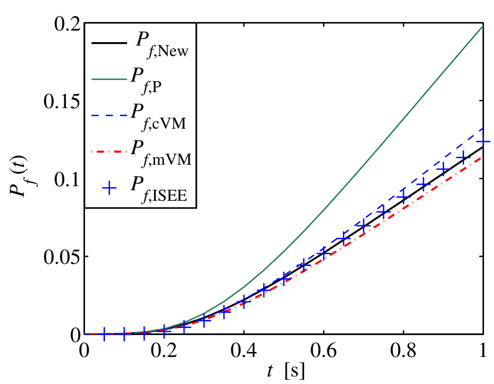

Fig. 7. FPFP for a linear elastic SDOF system with natural period $T = 0.1$ s subjected to KT base excitation from at-rest initial conditions ($\xi = 0.035$, $\zeta = 3$)

Table 8. FPFP Computed at $t = 20.0$ s for a Linear Elastic SDOF System with Natural Period $T = 0.1$ s Subjected to KT Base Excitation Time Modulated by a Shinozuka-Sato Function

ζ	ξ	ISEE	P	ε (%)	cVM	ε (%)	mVM	ε (%)	New	ε (%)
2	0.01	8.74E − 01	1.00E + 00	14.45	9.93E − 01	13.67	9.73E − 01	11.35	9.31E − 01	6.52
	0.05	9.92E − 01	1.00E + 00	0.80	9.99E − 01	0.75	9.98E − 01	0.62	9.99E − 01	0.71
	0.10	9.90E − 01	1.00E + 00	1.05	1.00E + 00	1.02	9.99E − 01	0.97	1.00E + 00	1.03
	0.50	9.93E − 01	1.00E + 00	0.64	9.99E − 01	0.55	9.98E − 01	0.49	9.99E − 01	0.63
3	0.01	1.67E − 01	5.46E − 01	226.13	2.94E − 01	75.81	2.28E − 01	36.07	1.76E − 01	5.23
	0.05	3.86E − 01	5.04E − 01	30.35	3.85E − 01	−0.50	3.45E − 01	−10.85	3.68E − 01	−4.91
	0.10	4.45E − 01	4.74E − 01	6.51	3.94E − 01	−11.59	3.66E − 01	−17.74	4.05E − 01	−9.08
	0.50	3.59E − 01	3.63E − 01	1.01	3.29E − 01	−8.45	3.18E − 01	−11.37	3.48E − 01	−3.04
4	0.01	6.87E − 03	1.81E − 02	163.96	9.66E − 03	40.61	7.34E − 03	6.85	5.58E − 03	−18.83
	0.05	1.40E − 02	1.61E − 02	15.02	1.26E − 02	−9.87	1.13E − 02	−19.60	1.20E − 02	−14.10
	0.10	1.44E − 02	1.48E − 02	3.12	1.27E − 02	−11.61	1.18E − 02	−17.67	1.30E − 02	−9.28
	0.50	1.03E − 02	1.04E − 02	0.52	9.74E − 03	−5.81	9.48E − 03	−8.24	1.01E − 02	−1.82
			ε_{max}%	226.13	ε_{max}%	75.81	ε_{max}%	36.07	ε_{max}%	5.23
			ε_{min}%	0.52	ε_{min}%	−11.61	ε_{min}%	−19.60	ε_{min}%	−18.83
			$\mu_{\|\varepsilon\|}$%	38.63	$\mu_{\|\varepsilon\|}$%	15.02	$\mu_{\|\varepsilon\|}$%	11.82	$\mu_{\|\varepsilon\|}$%	6.27

Table 9. FPFP Computed at $t = 20.0$ s for a Linear Elastic SDOF System with Natural Period $T = 0.5$ s Subjected to KT Base Excitation Time Modulated by a Shinozuka-Sato Function

ζ	ξ	ISEE	P	ε (%)	cVM	ε (%)	mVM	ε (%)	New	ε (%)
2	0.01	4.56E − 01	9.80E − 01	114.93	6.23E − 01	36.55	4.73E − 01	3.74	4.19E − 01	−8.12
	0.05	6.80E − 01	9.40E − 01	38.16	7.44E − 01	9.35	6.57E − 01	−3.37	6.43E − 01	−5.45
	0.10	8.12E − 01	9.36E − 01	15.23	8.11E − 01	−0.14	7.55E − 01	−7.08	7.90E − 01	−2.72
	0.50	9.45E − 01	9.33E − 01	−1.36	9.02E − 01	−4.58	8.89E − 01	−5.94	9.31E − 01	−1.52
3	0.01	4.83E − 02	2.03E − 01	319.65	6.84E − 02	41.54	4.64E − 02	−4.05	4.86E − 02	0.47
	0.05	8.21E − 02	1.50E − 01	82.93	8.82E − 02	7.39	7.18E − 02	−12.50	8.00E − 02	−2.57
	0.10	1.09E − 01	1.47E − 01	35.21	1.03E − 01	−5.02	9.01E − 02	−17.09	1.10E − 01	1.12
	0.50	1.45E − 01	1.45E − 01	−0.47	1.30E − 01	−10.54	1.26E − 01	−13.55	1.40E − 01	−3.71
4	0.01	1.68E − 03	5.25E − 03	212.60	2.04E − 03	21.30	1.39E − 03	−17.13	1.48E − 03	−11.54
	0.05	2.58E − 03	3.77E − 03	46.12	2.50E − 03	−3.21	2.07E − 03	−19.82	2.36E − 03	−8.43
	0.10	3.17E − 03	3.68E − 03	16.13	2.86E − 03	−9.89	2.54E − 03	−19.97	3.10E − 03	−2.20
	0.50	3.60E − 03	3.61E − 03	0.44	3.40E − 03	−5.46	3.32E − 03	−7.69	3.55E − 03	−1.45
			ε_{max}%	319.65	ε_{max}%	41.54	ε_{max}%	−4.05	ε_{max}%	1.12
			ε_{min}%	−0.47	ε_{min}%	−10.54	ε_{min}%	−19.97	ε_{min}%	−11.54
			$\mu_{\|\varepsilon\|}$%	73.60	$\mu_{\|\varepsilon\|}$%	12.91	$\mu_{\|\varepsilon\|}$%	10.99	$\mu_{\|\varepsilon\|}$%	4.11

Table 10. FPFP Computed at $t = 20.0$ s for a Linear Elastic SDOF System with Natural Period $T = 1.0$ s Subjected to KT Base Excitation Time Modulated by a Shinozuka-Sato Function

ζ	ξ	ISEE	P	ε (%)	cVM	ε (%)	mVM	ε (%)	New	ε (%)
2	0.01	3.94E − 01	9.21E − 01	133.80	4.96E − 01	25.79	3.72E − 01	−5.54	3.26E − 01	−17.28
	0.05	4.94E − 01	7.85E − 01	58.96	5.59E − 01	13.11	4.85E − 01	−1.92	4.85E − 01	−1.87
	0.10	6.03E − 01	7.57E − 01	25.52	6.07E − 01	0.59	5.55E − 01	−8.05	5.97E − 01	−1.04
	0.50	7.64E − 01	7.54E − 01	−1.25	7.18E − 01	−6.00	7.04E − 01	−7.86	7.54E − 01	−1.30
3	0.01	4.00E − 02	1.37E − 01	242.28	4.89E − 02	22.37	3.41E − 02	−14.59	3.44E − 02	−13.94
	0.05	4.89E − 02	8.53E − 02	74.40	5.34E − 02	9.16	4.47E − 02	−8.70	4.95E − 02	1.20
	0.10	6.02E − 02	7.86E − 02	30.57	5.83E − 02	−3.21	5.21E − 02	−13.57	6.10E − 02	1.27
	0.50	7.82E − 02	7.80E − 02	−0.14	7.18E − 02	−8.07	7.01E − 02	−10.26	7.52E − 02	−3.79
4	0.01	1.34E − 03	3.39E − 03	153.24	1.44E − 03	7.35	1.02E − 03	−24.13	1.08E − 03	−18.98
	0.05	1.51E − 03	2.07E − 03	37.03	1.47E − 03	−2.66	1.25E − 03	−16.89	1.50E − 03	−0.46
	0.10	1.69E − 03	1.90E − 03	12.40	1.55E − 03	−8.00	1.41E − 03	−16.20	1.70E − 03	0.46
	0.50	1.84E − 03	1.88E − 03	2.34	1.80E − 03	−1.97	1.78E − 03	−3.44	1.82E − 03	−1.19
			ε_{max}%	242.28	ε_{max}%	22.37	ε_{max}%	−3.44	ε_{max}%	1.27
			ε_{min}%	−0.14	ε_{min}%	−8.07	ε_{min}%	−24.13	ε_{min}%	−18.98
			$\mu_{\|\varepsilon\|}$%	64.33	$\mu_{\|\varepsilon\|}$%	9.02	$\mu_{\|\varepsilon\|}$%	10.93	$\mu_{\|\varepsilon\|}$%	5.23

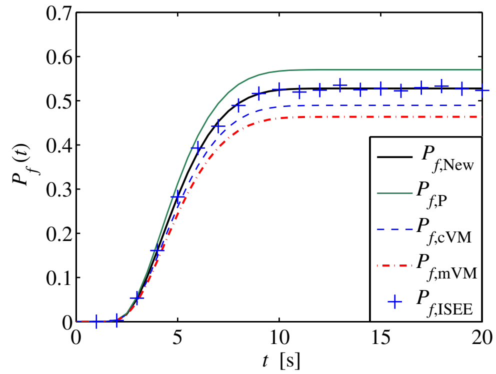

Fig. 8. FPFP for a linear elastic SDOF system with natural period $T = 1.0$ s subjected to KT base excitation with a Shinozuka-Sato modulating function ($\xi = 0.22$, $\zeta = 2.2$)

the one achieved in the case of linear SDOF systems subjected to WN excitation from at-rest initial conditions.

Fig. 7 plots the comparison of the FPFP estimates obtained using the ISEE method and the P, cVM, mVM, and New approximations for a linear SDOF system with $T = 0.1$ s and $\xi = 0.035$, for a normalized threshold value of $\zeta = 3.0$. It is observed that the P approximation significantly overestimates the FPFP, the two Vanmarcke approximations provide an upper and a lower bound of the FPFP, and the New approximation is in very good agreement with the ISEE results.

Results for Linear SDOF Systems Subjected to a KT Excitation Modulated in Time by a Shinozuka-Sato Function

Tables 8–10 compare the FPFP values computed using the ISEE method and the four analytical approximations considered for linear SDOF systems with $T = 0.1$, 0.5, and 1.0 s, respectively, subjected to a KT excitation modulated in time by a Shinozuka-Sato function. The observations regarding the absolute and relative accuracy of the analytical approximations are very similar to the observations made in the previous cases. In particular, the New approximation is much more accurate than the P approximation and significantly more accurate than both the cVM and mVM approximations. Fig. 8 plots the comparison of the FPFP estimates obtained using the ISEE method and the P, cVM, mVM, and New approximations for a linear SDOF system with $T = 1.0$ s and $\xi = 0.22$ for a normalized threshold value of $\zeta = 2.2$. In this case, the P and cVM approximations provide an upper and a lower bound, respectively, for the FPFP, whereas the New approximation is in excellent agreement with the ISEE results.

Conclusion

The classical first-passage reliability problem for linear elastic SDOF oscillators subjected to stationary and nonstationary Gaussian excitations is explored. Existing analytical approximations (e.g., Poisson, classical Vanmarcke, and modified Vanmarcke approximations) are reviewed and compared in terms of absolute and relative accuracy in estimating the time-variant FPFP.

A new analytical approximation of the FPFP for linear SDOF systems is proposed by modifying the classical Vanmarcke hazard function. This new approximation is verified by comparing its analytical estimates of the FPFP with the corresponding results obtained using existing analytical approximations and the ISEE method for a wide range of damping ratios, natural periods, and threshold levels. Different types of stochastic input excitations are considered, including white and nonwhite spectra, with a sudden application of the loads (i.e., with a unit step time-modulating function) or with a load modulated in time by using a Shinozuka-Sato function. The following conclusions are reached:

1. The Poisson approximation generally overestimates (sometimes by a large factor) the FPFP;
2. The two Vanmarcke approximations are significantly more accurate than Poisson's approximation;
3. The classical Vanmarcke approximation tends to overestimate the FPFP;
4. The modified Vanmarcke approximation tends to underestimate the FPFP;
5. The relative accuracy of the two Vanmarcke approximations is different on a case-by-case basis; and
6. The newly proposed analytical approximation of the hazard function yields significantly improved estimates of the FPFP when compared with Poisson, classical Vanmarcke, and modified Vanmarcke approximations.

This paper focuses on nonstationary excitations modeled as separable nonstationary processes. Research is ongoing to extend the proposed improved approximation of the time-variant FPFP to linear elastic systems subjected to more general fully nonstationary excitation models (e.g., processes with time-varying amplitude and frequency content obtained as the summation of separable nonstationary processes).

The new analytical approximation of the time-variant FPFP for linear elastic SDOF systems subjected to stationary and/or nonstationary Gaussian excitations provides an extremely valuable tool for validating, in the linear range, numerical methods used to estimate failure probabilities in more general cases, such as those involving nonlinear structural behavior and non-Gaussian excitations. This proposed analytical approximation is also very useful in applications requiring numerous repeated computations of the failure probability such as parametric studies and design optimization because its computational cost is only a very small fraction of the computational cost associated with simulation techniques. Finally, this new analytical approximation represents an important first step toward the development of a more general approximation of the time-variant FPFP for linear and nonlinear multi-degree-of-freedom systems subjected to stationary and/or nonstationary excitations.

Appendix. FPFP Dependence on Normalized Time

The time-variant FPFP for a linear SDOF system subjected to a WN excitation from at-rest initial conditions can be expressed as a function of only the normalized time, normalized threshold, and damping ratio. This fact can be shown by comparing the ISEE results for the FPFP obtained for SDOF systems with different natural periods (see Fig. 9) and can be expressed mathematically as

$$P_{f,|X|}(x_{\lim}, t) = 1 - \exp\left\{-\int_0^t h_{|X|}(x_{\lim}, \tau)\cdot d\tau\right\}$$
$$= 1 - \exp\left\{-\int_0^{t_0} \bar{h}_{|X|}(\zeta, \tau_0)\cdot d\tau_0\right\} = \bar{P}_{f,|X|}(\zeta, t_0) \tag{15}$$

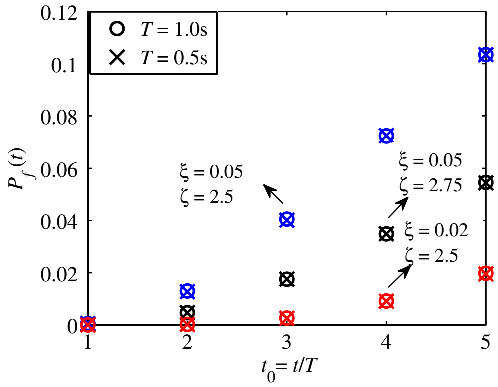

Fig. 9. Comparison of FPFP for linear elastic SDOF systems with $T = 1.0$ and 0.5 s subjected to WN base excitation from at-rest initial conditions

In this appendix, a superposed bar indicates a function that depends on time and natural period only through the normalized time t_0. Eq. (15) implies that, for any natural period T, the following relation holds:

$$h_{|X|}(x_{\lim}, t) = \frac{1}{T} \cdot \bar{h}_{|X|}(\zeta, t_0) \tag{16}$$

It can be demonstrated that the analytical approximations of the hazard function given in Eqs. (2)–(5) satisfy Eq. (16). It is sufficient to prove that Eq. (16) is satisfied by the newly proposed hazard function given in Eq. (5). In Barbato and Conte (2008), it was shown that the bandwidth parameter can be expressed as a function of the normalized time [i.e., $q_X(t) = \bar{q}_X(t_0)$]. This relation also implies that $C_1(\xi, \zeta, t) = \bar{C}_1(\xi, \zeta, t_0)$ and $C_2(\xi, \zeta, t) = \bar{C}_2(\xi, \zeta, t_0)$. From the closed-form solutions for the second-order statistics of the displacement and velocity response of a linear SDOF system subjected to a WN excitation with at-rest initial conditions (Lutes and Sarkani 2004), it can also be shown that

$$\sigma_X^2(t) = \sigma_{X\infty}^2 \cdot \bar{\sigma}_X^2(t_0) \tag{17}$$

$$\sigma_{\dot{X}}^2(t) = \frac{\sigma_{X\infty}^2}{T^2} \cdot \bar{\sigma}_{\dot{X}}^2(t_0) \tag{18}$$

$$K_{X\dot{X}}(t) = \frac{\sigma_{X\infty}^2}{T} \cdot \bar{K}_{X\dot{X}}(t_0) \tag{19}$$

in which $K_{X\dot{X}}(t)$ = cross covariance of the displacement and velocity responses of the linear SDOF system. The relation $\omega_0 \cdot t = 2 \cdot \pi \cdot t_0$ is used in Eqs. (17)–(19). Thus, the following relations are also valid:

$$\rho_{X\dot{X}}(t) = \frac{K_{X\dot{X}}(t)}{\sigma_X(t) \cdot \sigma_{\dot{X}}(t)} = \frac{(\sigma_{X\infty}^2/T) \cdot \bar{K}_{X\dot{X}}(t_0)}{(\sigma_{X\infty}^2/T) \cdot \sqrt{\bar{\sigma}_X(t_0) \cdot \bar{\sigma}_{\dot{X}}(t_0)}} = \bar{\rho}_{X\dot{X}}(t_0) \tag{20}$$

$$r(t) = \frac{\rho_{X\dot{X}}(t)}{\sqrt{2 \cdot [1 - \rho_{X\dot{X}}^2(t)]}} \cdot \left[\frac{x_{\lim}}{\sigma_X(t)}\right] = \frac{\bar{\rho}_{X\dot{X}}(t_0)}{\sqrt{2 \cdot [1 - \bar{\rho}_{X\dot{X}}^2(t_0)]}} \cdot \left[\frac{\zeta}{\bar{\sigma}_X(t_0)}\right] = \bar{r}(t_0) \tag{21}$$

$$\begin{aligned} \upsilon_{|X|}(x_{\lim}, t) &= \frac{\sigma_{\dot{X}}(t) \cdot \sqrt{1 - \rho_{X\dot{X}}^2(t)}}{\pi \cdot \sigma_X(t)} \cdot \exp\left(-\frac{r(t)}{\rho_{X\dot{X}}^2(t)}\right) \\ &\quad \cdot \{1 + \sqrt{\pi} \cdot r(t) \cdot \exp[r^2(t)] \cdot \text{erfc}[-r(t)]\} \\ &= \frac{\bar{\sigma}_{\dot{X}}(t_0) \cdot \sqrt{1 - \bar{\rho}_{X\dot{X}}^2(t_0)}}{T \cdot \pi \cdot \bar{\sigma}_X(t_0)} \cdot \exp\left(-\frac{\bar{r}(t_0)}{\bar{\rho}_{X\dot{X}}^2(t_0)}\right) \\ &\quad \cdot \{1 + \sqrt{\pi} \cdot \bar{r}(t_0) \cdot \exp[\bar{r}^2(t_0)] \cdot \text{erfc}[-\bar{r}(t_0)]\} \\ &= \frac{1}{T} \cdot \bar{\upsilon}_{|X|}(\zeta, t_0) \end{aligned} \tag{22}$$

Finally, the following relation is obtained:

$$\begin{aligned} h_{\text{New},|X|}(x_{\lim}, t) &= \upsilon_{|X|}(x_{\lim}, t) \cdot \exp[C_2(\xi, \zeta, t)] \\ &\quad \cdot \frac{1 - \exp\{-\sqrt{\pi/2} \cdot [q_X(t)]^{C_1(\xi,\zeta,t)} \cdot [x_{\lim}/\sigma_X(t)\}}{1 - \exp\{-\frac{1}{2} \cdot [x_{\lim}/\sigma_X(t)]^2\}} \\ &= \frac{1}{T} \bar{\upsilon}_{|X|}(\zeta, t_0) \cdot \exp[\bar{C}_2(\xi, \zeta, t_0)] \\ &\quad \cdot \frac{1 - \exp\{-\sqrt{\pi/2} \cdot [\bar{q}_X(t_0)]^{\bar{C}_1(\xi,\zeta,t_0)} \cdot [\zeta/\bar{\sigma}_X(t_0)]\}}{1 - \exp\{-\frac{1}{2} \cdot [\zeta/\bar{\sigma}_X(t_0)]^2\}} \\ &= \frac{1}{T} \cdot \bar{h}_{\text{New},|X|}(\zeta, t_0) \end{aligned} \tag{23}$$

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